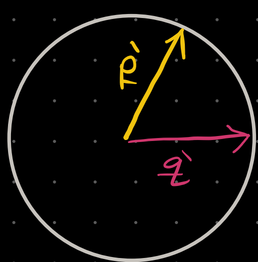
Classical Fidelity

A measure of the similarity between two distributions

$$F(p, q) = \sum_i \sqrt{p_i q_i}$$

i.e. inner product between $\sqrt{\ }$ of distribution vectors which lie on unit sphere.



$$\grave{p} = (\sqrt{p_0}, \sqrt{p_1}, \dots)$$

$$\grave{q} = (\sqrt{q_0}, \sqrt{q_1}, \dots)$$

Properties

- $F(p, p) = 1 = \text{Max } F(p, q)$
- $F(p, q) = 0$ if distributions have no overlapping support
- $F(p, q) \geq 0$

Mixed State Fidelity

$$F(\rho, \sigma) = \min_{\{M_i\}} \sum_i \sqrt{\mathrm{Tr}(\rho M_i)\,\mathrm{Tr}(\sigma M_i)}$$

i.e. the POVM that best discriminates two states as measured via the classical fidelity.

equivalent to $\Longrightarrow$

$$F(\rho, \sigma) = \mathrm{Tr}\left(\sqrt{\rho^{1/2} \sigma \rho^{1/2}}\right)$$

proof Fuchs - 1994

Outlined in problem sheet!

Case 1: ρ & σ commute $\rightarrow$ $\rho = \sum_i r_i |i\rangle\langle i|$ $\sigma = \sum_i s_i |i\rangle\langle i|$

$$F(\rho,\sigma) = Tr\left(\sqrt{\sum_i r_i s_i |i\rangle\langle i|}\right)$$
$$= Tr\left(\sum_i \sqrt{r_i s_i}\, |i\rangle\langle i|\right)$$
$$= \sum_i \sqrt{r_i s_i}$$
$$= F(\underline{r}, \underline{s})$$

Classical fidelity

Case 2: σ, $\rho = |\psi\rangle\langle\psi|$ $\qquad (|\psi\rangle\langle\psi|)^2 = \rho \Rightarrow \rho^{1/2} = |\psi\rangle\langle\psi|$

$$F(\rho,\sigma) = Tr\left(\sqrt{|\psi\rangle\langle\psi|\sigma|\psi\rangle\langle\psi|}\right)$$
$$= Tr\left(\sqrt{\langle\psi|\sigma|\psi\rangle}\ \underbrace{\sqrt{|\psi\rangle\langle\psi|}}_{|\psi\rangle\langle\psi|}\right)$$
$$= \sqrt{\langle\psi|\sigma|\psi\rangle}$$

Fidelity between pure and mixed state is equal to the overlap

Case 2b. $\sigma = |\phi\rangle\langle\phi|$ $\qquad F(\rho,\sigma) = |\langle\psi|\phi\rangle|$

Note the lack of mod square here $\rightarrow$ this is a matter of convention I'm following N&C here.

General case? Operational interpretation provided by Uhlmann's Theorem.

Uhlmann's Theorem:

max over all possible purifications of ψ & φ

$$F(\rho,\sigma) = \max_{|\psi\rangle, |\varphi\rangle} |\langle\psi|\varphi\rangle|$$

where $\rho_S = \mathrm{Tr}_R(|\psi\rangle\langle\psi|_{RS})$ & $\sigma_S = \mathrm{Tr}_R(|\psi\rangle\langle\psi|_{RS})$

proof - exercise sheet this week.

Data processing inequality also holds here

$$F(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \geq F(\rho,\sigma)$$